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'Spin–spin' Hall effect in two dimensional electron systems with spin–orbit interaction

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Abstract
We calculate spin-Hall conductivities in two dimensional electron systems with Rashba spin–orbit interaction. The salient feature is that, apart from the usual spin-Hall conductivity σ^{z}_{xy}, which corresponds to the induction of out-of-plane spin current due to the application of transverse charge current, there is a novel spin-Hall conductivity $\sigma^{\perp}_{xy}$, which arises due to the induction of transverse spin-polarized current in the transverse direction by the application of in-plane spin-polarized current. This phenomenon, which we call the 'spin–spin' Hall effect, is a spin analogue of the conventional Hall effect, but with no magnetic field. This contribution may be understood through the spin-diffusive equation.

1. Introduction

Following the pioneering proposal of a device called the spin field effect transistor (SFET) by Datta and Das [1], there has been growing interest in the field of spintronics [2], which is the science of coherent manipulation of spin, because of its potential applications in spin-memory and quantum computing devices [3]. This device consists of a two dimensional electron system (2DES), controlled by gate voltage in a semiconductor heterostructure with spin-polarized contacts. A spin entered from a spin-polarized source in the 2DES precesses due to the Rashba spin–orbit (SO) interaction arising from the lack of structural inversion symmetry in the semiconductor heterostructures. The spin of the electrons can be made to transport coherently and the spin-polarized drain whose spin polarization is parallel to that of the source can detect the transportation of spin. The spin current can be manipulated by controlling gate voltage since the Rashba SO coupling depends on it. However, the momentum relaxation due to elastic scattering of the electrons from impurity potential leads to spin relaxation and thereby destroys spin coherence. Although SFET has not been achieved yet, the modulation of SO coupling by gate voltage has been observed in InGaAs/InAlAs and GaAs/AlGaAs heterostructures [4–6]. This motivates us to study the system of electrons with SO interactions.

A remarkable consequence of the SO interaction is the spin-Hall effect where an electric field induces a transverse out-of-plane spin current and thereby a spin imbalance takes place between two opposite edges of the sample. Ever since the prediction of the intrinsic dissipationless spin-Hall effect in hole-doped semiconductor systems [7], this effect has drawn intense theoretical as well as experimental activities. Although Sinova *et al* [8] have predicted universal (independent of the strength of Rashba spin–orbit interaction) dc spin-Hall conductivity (SHC) $\sigma_{xy}^{z} = e/8\pi$ in a pure 2DES, the issue of SHC in the presence of non-magnetic impurities remains highly controversial. A number of analytical works [9–11] based on the Kubo formula and quantum kinetic equations suggest that σ_{xy}^{z} vanishes for any amount of elastic disorder in the diffusive transport regime. This dramatic vanishing of SHC in the presence of disorder is argued to be a more generic phenomenon [12] as σ_{xy}^{z} also vanishes for any amount of magnetic field in a pure system. In contrast, experiments [15–17] seem to suggest that there is nonequilibrium spin accumulation in the edges, transverse to the direction of application of charge current. Numerical studies [13] based on the Landauer–Buttiker approach in mesoscopic systems with leads cannot shed any light on SHC in the thermodynamic limit because of the possibility of edge effects near the contacts. However, a numerical calculation based on the Kubo formula in the lattice model [14] suggests that SHC is finite in the presence of disorder, while it vanishes in the thermodynamic limit. Nomura *et al* [18] have numerically shown that the vanishing of intrinsic σ_{xy}^{z} is peculiar to the linear-momentum Rashba model which has been considered in the previous studies. They have shown that for the Rashba model with cubic momentum, which is the case for a two dimensional hole gas, the intrinsic σ_{xy}^{z} is non-zero and consequently intrinsic edge-spin accumulation takes place in the systems of two dimensional hole gases [16, 19]. On the other hand, the edge-spin accumulation observed [15, 17] in the system of a two dimensional electron gas where the Rashba model with linear momentum is important, is argued to be extrinsic by the 'skew-scattering' mechanism [20–22]. While the issue of intrinsic versus extrinsic spin-Hall effect is not clear yet, we show below that the linear Rashba model provides a different but robust kind of spin-Hall effect in the two dimensional electron systems.

In this paper, we calculate the spin-Hall conductivities in the charge-spin space using the Kubo formula in section 3. A charge current induces an out-of-plane component of spin current in the transverse direction. This usual ac spin-Hall conductivity σ_{xy}^{z} is found to be the same as in a kinetic equation approach [10]. It however vanishes in the dc limit [9–11]. We call this phenomenon the 'charge-spin' Hall effect. Apart from this, we find that an in-plane spin-polarized current can induce a transverse spin-polarized current along the transverse spatial direction. This phenomenon, which we call the 'spin–spin' Hall effect, is a spin analogue of the conventional charge Hall effect, but in the absence of magnetic field. The relevant Hall conductivity of this phenomenon $\sigma_{xy}^{\perp}$ does not vanish in the dc limit. We attribute this phenomenon to the diffusion of spin s_y along the y-direction due to a source causing conventional s_x spin diffusion along the x-direction. The corresponding diffusive equation is derived in section 4. The detailed derivation of the diffusive propagator is available in the appendix A. In section 5, the spin–spin Hall conductivity has been argued to be robust. The relevance of other types of spin–spin Hall conductivity have also been discussed. The consequence of Dresselhaus SO coupling along with the Rashba coupling on spin–spin Hall conductivities has been discussed in appendix B.

2. Hamiltonian and Green's functions

A system of two dimensional non-interacting electrons with Rashba spin–orbit interaction in the presence of non-magnetic impurities from which the electrons scatter elastically can be described by the Hamiltonian

$$H = H_0 + V(\mathbf{r}); \qquad H_0 = \frac{\mathbf{p}^2}{2m} + \lambda\hat{\boldsymbol{\eta}}\cdot\mathbf{p}, \tag{1}$$

where $\mathbf{p} = -\mathrm{i}\nabla$ is the momentum operator, m is the effective band mass of electrons, λ is the parameter for the strength of the Rashba spin–orbit interaction, $\hat{\boldsymbol{\eta}} = \mathbf{n} \times \hat{\boldsymbol{\sigma}}$ is a spin operator with $\hat{\boldsymbol{\sigma}}$ being the Pauli matrices, $\mathbf{n}$ is the unit vector perpendicular to the plane of the system, and $V(\mathbf{r})$ is the disorder potential. The eigenvalues of H_0 are $\epsilon_{\mathbf{k}}^{s} = k^2/2m + s\lambda k$, where k is the magnitude of the wavevector $\mathbf{k}$ of the electrons and $s = \pm$ is the index for spin-split subband s. The corresponding eigenstates are given by

$$\psi_{\mathbf{k}}^{s}(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi_{\mathbf{k}}/2} \\ s\mathrm{e}^{-\mathrm{i}\chi_{\mathbf{k}}/2} \end{pmatrix}, \tag{2}$$

where $\chi_{\mathbf{k}} = \pi/2 - \theta$ with θ being the angle of $\mathbf{k}$ with the x-axis. The Fermi momenta and the density of states at the Fermi energy, ϵ_{F}, in the two subbands are $k_{\mathrm{F}}^{s} = \sqrt{k_{\mathrm{F}}^2 + m^2\lambda^2} - sm\lambda$ and $\nu^{s} = \nu(1 - s\frac{m\lambda}{\sqrt{k_{\mathrm{F}}^2 + m^2\lambda^2}})$ respectively, where $\nu = m/2\pi$ is the density of states for each spin direction and $k_{\mathrm{F}} = \sqrt{2m\epsilon_{\mathrm{F}}}$ is the Fermi momentum in the absence of SO interaction. The charge and spin currents are $\hat{\mathbf{j}}_{\alpha} = e_{\alpha}\frac{1}{2}\left\{\hat{\sigma}_{\alpha}, \hat{\mathbf{v}}_{\mathbf{k}}\right\}$ where $\hat{\mathbf{v}}_{\mathbf{k}} = \frac{\mathbf{k}}{m}\hat{\sigma}_0 + \lambda\hat{\boldsymbol{\eta}}$ is the velocity and $\hat{\sigma}_0$ is the unit matrix. Here $\alpha = 0$–3 represents charge and three spin directions respectively and $e_0 = e$ (charge of the electrons), $e_1 = e_2 = e_3 = 1/2$ (spin of the electrons).

We next assume random disorder potential with the configurational average such that $\langle V(\mathbf{r})\rangle = 0$ and $\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \gamma\delta(\mathbf{r} - \mathbf{r}')$. The elastic lifetime of the electrons will then be $\tau = 1/(2\pi\gamma\nu)$. Therefore, the retarded (advanced) Green's function for the electrons with energy ϵ and wavevector $\mathbf{k}$ can be expressed as

$$\hat{\mathcal{G}}^{R,A}(\mathbf{k}, \epsilon) = \frac{1}{2}\sum_{s=\pm} \frac{\hat{\sigma}_0 + s\hat{\eta}_{\mathbf{k}}}{\epsilon - \xi_{\mathbf{k}}^{s} \pm \mathrm{i}/2\tau} \equiv \sum_{s=\pm} \hat{\mathcal{G}}_{s}^{R,A}(\mathbf{k}, \epsilon), \tag{3}$$

where $\hat{\eta}_{\mathbf{k}} = \hat{\boldsymbol{\eta}}\cdot\mathbf{k}/k$ is the projection of the spin operator into the direction of $\mathbf{k}$ and $\xi_{\mathbf{k}}^{s} = \epsilon_{\mathbf{k}}^{s} - \epsilon_{\mathrm{F}}$.

3. Hall conductivities

The spin-Hall conductivities σ_{xy}^{z}, $\sigma_{xy}^{\perp}$, and $\sigma_{xy}^{\parallel}$ correspond to transverse induction of out-of-plane (z-axis) spin current due to a charge current, transverse (in-plane) spin current due to a spin-polarized current, and parallel spin current due to a spin-polarized current respectively. These conductivities can be obtained using the Kubo formula:

$$\sigma_{xy}^{z}(\omega) = \frac{1}{4\pi}\left(\Pi_{xy}^{03} + \Pi_{xy}^{30}\right), \tag{4}$$

$$\sigma_{xy}^{\perp}(\omega) = \frac{1}{4\pi}\left(\Pi_{xy}^{12} + \Pi_{xy}^{21}\right), \tag{5}$$

$$\sigma_{xy}^{\parallel}(\omega) = \frac{1}{4\pi}\left(\Pi_{xy}^{11} + \Pi_{xy}^{22}\right), \tag{6}$$

where retarded off-diagonal current–current correlation functions can be written as

$$\Pi_{xy}^{\alpha\beta}(\omega) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\,\mathrm{Tr}\left(\hat{j}_{\alpha}^{x}(\mathbf{k})\hat{\mathcal{G}}^{A}(\mathbf{k}, 0)\left[\hat{j}_{\beta}^{y}(\mathbf{k}) + \hat{J}_{\beta}^{y}(\omega)\right]\hat{\mathcal{G}}^{R}(\mathbf{k}, \omega)\right). \tag{7}$$

Here the current $\hat{\mathbf{j}}_{\beta}(\mathbf{k})$ corresponds to the bare vertex. This vertex gets corrected through $\hat{\mathbf{J}}_{\beta}$, which is the self-consistent solution of the equation

$$\hat{\mathbf{J}}_{\beta}(\omega) = \gamma\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\hat{\mathcal{G}}^{A}(\mathbf{k}', 0)\left[\hat{\mathbf{j}}_{\beta}(\mathbf{k}') + \hat{\mathbf{J}}_{\beta}(\omega)\right]\hat{\mathcal{G}}^{R}(\mathbf{k}', \omega). \tag{8}$$

It describes summation over an infinite number of ladder diagrams in a diagrammatic approach.

To solve equation (8) and evaluate equation (7), we need to consider integrations, e.g.,

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \hat{\mathcal{G}}_s^A(\mathbf{k},0)\hat{\mathcal{G}}_{s'}^R(\mathbf{k},\omega)$$

which we calculate using the approximation

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \Rightarrow \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\left[\nu^s \int_{-\infty}^{\infty} \mathrm{d}\xi_{\mathbf{k}}^s \delta_{ss'} + \nu \int_{-\infty}^{\infty} \mathrm{d}\xi_{\mathbf{k}}(1-\delta_{ss'})\right] \tag{9}$$

i.e., the integrations are performed around momentum k_{F}^s for the intraband contribution while the interband contributions are obtained through the expansion around k_{F}, and the variable of integration for the latter is $\xi_{\mathbf{k}} = k^2/2m - \epsilon_{\mathrm{F}}$. This approximation is, however, neither biased towards any of the subbands nor negligent of their existence. This attention is important for our result. The self-consistent solutions are thus straightforward and we find

$$\hat{\mathbf{J}}_0(\omega) = -\lambda \frac{2\delta^2}{2\delta^2(1-2\mathrm{i}\omega\tau) - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)^2}\hat{\boldsymbol{\eta}} \tag{10}$$

$$\hat{\mathbf{J}}_1(\omega) = -\left(\frac{k_{\mathrm{F}}}{2m}\right)\frac{\delta}{4\delta^2 - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)}\hat{\sigma}_3\hat{\mathbf{x}} \tag{11}$$

$$\hat{\mathbf{J}}_2(\omega) = -\left(\frac{k_{\mathrm{F}}}{2m}\right)\frac{\delta}{4\delta^2 - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)}\hat{\sigma}_3\hat{\mathbf{y}} \tag{12}$$

$$\hat{\mathbf{J}}_3(\omega) = \left(\frac{k_{\mathrm{F}}}{2m}\right)\frac{\delta(1-\mathrm{i}\omega\tau)}{2\delta^2(1-2\mathrm{i}\omega\tau) - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)^2}\hat{\boldsymbol{\sigma}} \tag{13}$$

for an arbitrary value of the parameter $\delta = \lambda k_{\mathrm{F}}\tau$. While $\hat{\mathbf{J}}_0$ and $\hat{\mathbf{J}}_3$ are due to both the intraband and the interband contributions, $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ are due to the interband contribution only. This means $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ exist only because of the formation of two subbands. In fact, all of these vanish for $\lambda = 0$, in consistence with the fact that the charge transport time and the momentum relaxation time are the same (no vertex correction) in a two dimensional system with short range scatterers.

Using equations (4)–(13) we find the Hall conductivities:

$$\sigma_{xy}^z(\omega) = \left(\frac{e\delta^2}{2\pi}\right)\frac{\mathrm{i}\omega\tau}{2\delta^2(1-2\mathrm{i}\omega\tau) - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)^2}, \tag{14}$$

$$\sigma_{xy}^{\perp}(\omega) = -\left(\frac{\Delta}{16\pi}\right)\frac{4\delta^2}{[4\delta^2 - \mathrm{i}\omega\tau(1-\mathrm{i}\omega\tau)][1-\mathrm{i}\omega\tau]}, \tag{15}$$

$$\sigma_{xy}^{\parallel}(\omega) = 0. \tag{16}$$

The spin-Hall conductivity $\sigma_{xy}^z(\omega)$ in equation (14) is identical to those derived from the quantum kinetic equation by Mishchenko *et al* [10] and in an approach of the Kubo formula by Chalaev and Loss [11]. At $\omega = 0$, the vertex correction term dramatically cancels [9–11] the bare contribution in σ_{xy}^z. The vanishing of σ_{xy}^z even in the weak localization regime [11] warns of the presence of an exact property of the system under consideration. From a general argument, Chalaev and Loss [11] have shown σ_{xy}^z vanishes exactly for any finite amount of disorder. Since the spin is not conserved, the spin-Hall conductivity may not be directly related [12] to the observed [15–17] spin accumulation at the transverse edges.

The most important result in this paper is the finding of the conductivity $\sigma_{xy}^{\perp}(\omega)$. This suggests that an in-plane spin-polarized current induces a transverse spin current with in-plane spin polarization but perpendicular to the former. If the source of the spin current is polarized along the x-axis, then a detector, placed transverse to the direction of the applied current, with polarization along the $\pm y$-axis can detect this novel current. Magnetic semiconductors may be used for spin injection as well as detection of the spin-polarized currents as described by Fiederling *et al* [23] in a light-emitting diode. Further techniques [15–17] developed for the detection of spin-accumulation in the spin-Hall effect may also be useful for observing spin–spin-Hall effect. Unlike the conventional spin-Hall conductivity σ^z_{xy}, this new spin–spin Hall conductivity is finite at $\omega = 0$ and is given by

$$\sigma^{\perp}_{xy}(\omega = 0) = -\frac{\Delta}{16\pi}, \tag{17}$$

independent of the Rashba coupling parameter λ. However, from the equation (15) we see that $\sigma^{\perp}_{xy}(\omega = 0) \neq 0$ only when $\lambda \neq 0$. In other words, $\lambda \to 0$ before $\omega \to 0$ is logically the correct limiting procedure for dc $\sigma^{\perp}_{xy}$ in the absence of spin–orbit interaction. The bare and vertex correction contribution to $\sigma^{\perp}_{xy}$ in equation (17) are $-\frac{\Delta}{16\pi}\left(\frac{4\delta^2}{1+4\delta^2}\right)$ and $-\frac{\Delta}{16\pi}\left(\frac{1}{1+4\delta^2}\right)$ respectively. Clearly the dominant contribution is due to the vertex correction. The vanishing $\sigma^{\parallel}_{xy}$ in equation (16) implies zero parallel spin-polarized current in the transverse direction.

4. Spin diffusion

We now derive the spin-diffusive equation for understanding the above novel spin–spin Hall effect. The retarded charge and spin density correlation functions at frequency ω and momentum $\mathbf{q}$ can be written as

$$\chi_{\alpha\beta}(\mathbf{q}, \omega) = \frac{\mathrm{i}\omega}{2\pi}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\,\mathrm{Tr}\left[\hat{\sigma}_\alpha \hat{\mathcal{G}}^A(\mathbf{k}, 0)\hat{M}_\beta \hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q}, \omega)\hat{\sigma}_\beta\right] \tag{18}$$

where $\alpha, \beta = 0$–3 corresponding to charge and three spin directions, $\hat{\mathcal{G}}^R_\beta(\mathbf{k}, \epsilon) = \hat{\sigma}_\beta \hat{\mathcal{G}}^R(\mathbf{k}, \epsilon)\hat{\sigma}_\beta$, and $\hat{M}_\beta = \sum_{j=0}^{\infty} \hat{m}^{(j)}_\beta$ with $\hat{m}^{(0)}_\beta = \hat{\sigma}_0$ and

$$\hat{m}^{(j)}_\beta = \gamma \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}', 0)\hat{m}^{(j-1)}_\beta \hat{\mathcal{G}}^R_\beta(\mathbf{k}'+\mathbf{q}, \omega); \qquad j \geqslant 1. \tag{19}$$

It is very easy to check that the equation (18) includes infinite number of ladder diagrams for the vertex correction. Although the expression of $\chi_{\alpha\beta}$ (18) appears to be uncommon, we find it to be a very convenient way to express for all orders of diagrams since we need to maintain the order of matrices inside the trace. The expression of $\chi_{\alpha\beta}$ in (18) is formal. We however evaluate it for small q and ω such that $\omega\tau \ll 1$ and $q \ll k_{\mathrm{F}}$, and using the approximation in the expression (9) for the integration over electron momenta. We iteratively calculate the coefficients of the $\hat{\sigma}_\alpha$ in the $\hat{m}^{(j)}_\beta$ (19) and take the sum of geometrical series of the coefficients. The readers interested in the details of the calculation may look at appendix A.

Expressing $\hat{m}^{(1)}_\beta = \sum_{\alpha=0}^{3} m_{\beta\alpha}\hat{\sigma}_\alpha$ and

$$\chi_{\alpha\beta} = 2\mathrm{i}\nu\omega\tau\left(\mathcal{D}_{\alpha\beta} - \delta_{\alpha\beta}\right) \tag{20}$$

in a standard form, we find the inverse diffusion propagator to be

$$\mathcal{D}^{-1} = \begin{bmatrix} 1-m_{00} & -m_{11} & -m_{22} & 0 \\ -m_{01} & 1-m_{10} & \frac{m_{21}m_{32}}{1-m_{30}} & -\mathrm{i}m_{32} \\ -m_{02} & \frac{m_{12}m_{31}}{1-m_{30}} & 1-m_{20} & \mathrm{i}m_{31} \\ 0 & \mathrm{i}m_{12} & -\mathrm{i}m_{21} & 1-m_{30} \end{bmatrix} \tag{21}$$

where $m_{00} = 1 + \tau(\mathrm{i}\omega - D\mathbf{q}^2)$, $m_{10} = m_{20} = m_{00} - \tau/\tau^{\parallel}_s$, $m_{30} = m_{00} - \tau/\tau^{\perp}_s$ with diffusion constant $D = \Delta/m$, and $\tau^{\perp}_s = \tau(1+4\delta^2)/4\delta^2$ and $\tau^{\parallel}_s = \tau(1+4\delta^2)/2\delta^2$ being the out-of-plane and in-plane spin relaxation times respectively, $m_{01} = m_{11} = 2\mathrm{i}q_y\delta^3/k_{\mathrm{F}}(1+4\delta^2)$, $m_{02} = m_{22} = -(q_x/q_y)m_{01}$, $m_{12} = m_{32} = -4q_x\delta\Delta/k_{\mathrm{F}}(1+4\delta^2)^2$ and $m_{21} = m_{31} = -(q_y/q_x)m_{12}$.

We thus find $\mathcal{D}_{21}^{-1} = -2q_x q_y \Delta\tau/m(1+4\delta^2)^3$ for $\omega = 0$. Clearly $\mathcal{D}_{21}^{-1}$ and $\mathcal{D}_{12}^{-1}$ are non-zero for small δ, contrary to the recent result of Burkov *et al* [24]. (See appendix A for details.) This is because the spin relaxation provides an energy cut-off. These are indeed the responsible terms for driving transverse spin in the transverse direction as we shall show below. However, the other components of $\mathcal{D}^{-1}$ are in agreement with reference [24].

Transformation of the inverse diffusive propagator (21) into real space and time leads to the transport equations for spins with planar projection (neglecting gradients of charge and out-of-plane components of spin densities):

$$\frac{\partial S_x}{\partial t} = \left(D\nabla^2 - \frac{1}{\tau_s^{\parallel}}\right) S_x - \frac{2D}{(1+4\delta^2)^3}\partial_x\partial_y S_y + I_x \tag{22}$$

$$\frac{\partial S_y}{\partial t} = \left(D\nabla^2 - \frac{1}{\tau_s^{\parallel}}\right) S_y - \frac{2D}{(1+4\delta^2)^3}\partial_x\partial_y S_x + I_y \tag{23}$$

where S_x (S_y) is the spin density along the x (y) direction and the x (y) component of the spin current I_x (I_y) is injected into the system. Although the coefficients of $\partial_x\partial_y S_y$ in equation (22) and $\partial_y\partial_x S_x$ in equation (23) appear to be non-zero for $\delta = 0$, these coefficients in fact vanish strictly at $\delta = 0$. The spin diffusive equations (22) and (23) are valid for $\omega\tau \ll 4\delta^2$. Clearly, the injected spin current $I_x(I_y)$ not only transports the S_x (S_y) component of spin along the direction of the current but also drives the S_y (S_x) component of spin along the transverse direction. The finite dc spin–spin Hall conductivity (17) including sign arises as a manifestation of this fact.

5. Discussion and summary

Since, due to the precession, the spin is not conserved [12, 24], the definition of spin current is not unique. We, however, have considered the generally accepted definition of the spin-current operator, i.e. the symmetrization of the product of spin operator and the group velocity of electrons. Apart from this spin current, which depends on the translational motion of spin, there is a current associated with the rotational motion [25]. Albeit the definition of current is not unique, the arbitrariness lies only in the terms proportional to the SO coupling parameter λ. The kinetic parts [24] (which are proportional to the momentum of electrons and independent of λ) of the spin currents are, however, unique. The substantial contribution in $\sigma_{xy}^{\perp}$ which arises due to vertex correction (17) is entirely due to the kinetic part of the spin currents, and it is not proportional to any power of λ. We therefore believe that the 'spin–spin' Hall effect is robust in the system of a two dimensional electron gas with Rashba spin–orbit interaction.

We have calculated $\sigma_{xy}^{\perp}$ and $\sigma_{xy}^{\parallel}$, which suggest spin–spin Hall conductivities when the induced spin current has respective spin polarization perpendicular and parallel to the spin polarization of the applied spin current. These spin polarizations are in the plane of the 2DES. While the mechanism for the usual spin-Hall effect in this system is mainly the momentum dependent spin precession along with the spin relaxation, the mechanism behind the spin–spin Hall effect is spin diffusion. If the spin polarization of the applied spin current is out of the plane of the system, we find the corresponding components of off-diagonal current–current correlation functions (7) vanish: $\Pi_{xy}^{13} + \Pi_{xy}^{31} = \Pi_{xy}^{23} + \Pi_{xy}^{32} = \Pi_{xy}^{11} + \Pi_{xy}^{33} = \Pi_{xy}^{22} + \Pi_{xy}^{33} = 0$. In other words, the spin–spin Hall effect is possible *only* when the applied spin-current's polarization is in the plane of the system.

In summary, we predict that an in-plane spin-polarized current can induce a transverse spin-polarized current along the transverse direction. This phenomenon, which we call the 'spin–spin' Hall effect, is a spin analogue of the conventional Hall effect, but with no magnetic field. The reason for this phenomenon is the transverse spin diffusion along the transverse direction due to the application of a source causing conventional spin diffusion.

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Appendix A. Diffusive propagator

In this appendix, we present a detailed calculation of the charge–spin density correlation function, and hence the inverse diffusive propagator, and compare with the previous result [24]. The retarded charge–spin density correlation function can be represented by a sum of an infinite number of ladder diagrams:

$$\chi_{\alpha\beta}(\mathbf{q},\omega) = \sum_{j=0}^{\infty} \chi_{\alpha\beta}^{(j)}(\mathbf{q},\omega), \tag{A.1}$$

where (j) represents the order of the diagrams. Explicitly these are given by

$$\chi_{\alpha\beta}^{(0)}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\hat{\sigma}_\alpha \hat{\mathcal{G}}^A(\mathbf{k},0)\hat{\sigma}_\beta \hat{\mathcal{G}}^R(\mathbf{k}+\mathbf{q},\omega)\right] \tag{A.2}$$

$$\chi_{\alpha\beta}^{(1)}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\gamma\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2}\hat{\sigma}_\alpha \hat{\mathcal{G}}^A(\mathbf{k},0) \times \hat{\mathcal{G}}^A(\mathbf{k}_1,0)\hat{\sigma}_\beta \hat{\mathcal{G}}^R(\mathbf{k}_1+\mathbf{q},\omega)\hat{\mathcal{G}}^R(\mathbf{k}+\mathbf{q},\omega)\right] \tag{A.3}$$

and so on. In the expression of $\chi_{\alpha\beta}$, the contribution at $\omega = 0$ and $q = 0$ for diagonal terms has been ignored. It is crucial to maintain the relative positions of the matrices inside the trace while performing the infinite series. To express all the orders in a compact form and to calculate those, we find it convenient to shift $\hat{\sigma}_\beta$ to the extreme right, but in doing so $\hat{\mathcal{G}}^R(\mathbf{k}+\mathbf{q},\omega)$ will be modified to $\hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q},\omega) = \hat{\sigma}_\beta\hat{\mathcal{G}}^R(\mathbf{k}+\mathbf{q},\omega)\hat{\sigma}_\beta$. We thus find

$$\chi_{\alpha\beta}^{(0)}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\hat{\sigma}_\alpha\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k},0)\hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q},\omega)\hat{\sigma}_\beta\right] \tag{A.4}$$

$$\chi_{\alpha\beta}^{(1)}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\hat{\sigma}_\alpha\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k},0)\left\{\gamma\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2} \times \hat{\mathcal{G}}^A(\mathbf{k}_1,0)\hat{\mathcal{G}}^R_\beta(\mathbf{k}_1+\mathbf{q},\omega)\right\}\hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q},\omega)\hat{\sigma}_\beta\right] \tag{A.5}$$

$$\chi_{\alpha\beta}^{(2)}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\hat{\sigma}_\alpha\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k},0)\left(\gamma\int\frac{\mathrm{d}\mathbf{k}_2}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}_2,0) \times \left\{\gamma\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}_1,0)\hat{\mathcal{G}}^R_\beta(\mathbf{k}_1+\mathbf{q},\omega)\right\}\hat{\mathcal{G}}^R_\beta(\mathbf{k}_2+\mathbf{q},\omega)\right)\hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q},\omega)\hat{\sigma}_\beta\right] \tag{A.6}$$

and so on. Defining

$$\hat{m}_\beta^{(j)}(\mathbf{q},\omega) = \gamma\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}',0)\hat{m}_\beta^{(j-1)}(\mathbf{q},\omega)\hat{\mathcal{G}}^R_\beta(\mathbf{k}+\mathbf{q},\omega); \qquad j \geqslant 1 \tag{A.7}$$

with $\hat{m}_\beta^{(0)} = \hat{\sigma}_0$, and summing over all orders (the lowest three orders given in equations (A.4)–(A.6)), we rewrite the retarded charge–spin density correlation function as

$$\chi_{\alpha\beta}(\mathbf{q},\omega) = \frac{\mathrm{i}\omega}{2\pi}\,\mathrm{Tr}\left[\hat{\sigma}_\alpha \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k},0)\left(\sum_{j=0}^{\infty}\hat{m}_\beta^{(j)}(\mathbf{q},\omega)\right)\hat{\mathcal{G}}_\beta^R(\mathbf{k}+\mathbf{q},\omega)\hat{\sigma}_\beta\right]. \tag{A.8}$$

Expressing

$$\begin{aligned}\hat{m}_\beta^{(1)}(\mathbf{q},\omega) &= \gamma\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}_1,0)\hat{\mathcal{G}}_\beta^R(\mathbf{k}_1+\mathbf{q},\omega)\\ &\equiv m_{\beta 0}\hat{\sigma}_0 + m_{\beta 1}\hat{\sigma}_1 + m_{\beta 2}\hat{\sigma}_2 + m_{\beta 3}\hat{\sigma}_3\end{aligned} \tag{A.9}$$

with β varying from 0 to 3, we obtain 16 components of $m_{\beta\alpha}$, which are $m_{00} = 1+\tau(\mathrm{i}\omega - D\mathbf{q}^2)$, $m_{10} = m_{20} = m_{00} - \frac{2\delta^2}{1+4\delta^2}$, $m_{30} = m_{00} - \frac{4\delta^2}{1+4\delta^2}$, $m_{01} = m_{11} = 2\mathrm{i}\frac{q_y}{k_F}\frac{\delta^3}{(1+4\delta^2)}$, $m_{02} = m_{22} = -\frac{q_x}{q_y}m_{01}$, $m_{12} = m_{32} = -\frac{q_x}{k_F}\frac{4\delta\Delta}{(1+4\delta^2)^2}$, $m_{21} = m_{31} = -\frac{q_y}{q_x}m_{12}$, $m_{03} = m_{33} = 0$; m_{13} and m_{23} are $\mathcal{O}\left(\frac{q^2}{k_F^2},\delta^2\right)$ and are very small compared to the others. Here diffusion constant $D = \frac{1}{2}v_F^2\tau$, dimensionless Rashba coupling strength $\delta = \lambda k_F\tau$ and $\Delta = \epsilon_F\tau$.

One has now to sum four infinite series of matrices, one for each value of β, to find out the matrix $\sum_{j=0}^{\infty}\hat{m}_\beta^{(j)}$, using (A.7) iteratively. We show the explicit calculation of $\sum_{j=0}^{\infty}\hat{m}_1^{(j)}$ below; others may be obtained following the same procedure.

From equation (A.7), we find

$$\begin{aligned}\hat{m}_1^{(2)} &= \gamma\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\hat{\mathcal{G}}^A(\mathbf{k}',0)\hat{m}_1^{(1)}(\mathbf{q},\omega)\hat{\mathcal{G}}_1^R(\mathbf{k}'+\mathbf{q},\omega)\\ &= \gamma\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\left[m_{10}\hat{\mathcal{G}}^A\hat{\sigma}_0\hat{\mathcal{G}}_1^R + m_{11}\hat{\mathcal{G}}^A\hat{\sigma}_1\hat{\mathcal{G}}_1^R + m_{12}\hat{\mathcal{G}}^A\hat{\sigma}_2\hat{\mathcal{G}}_1^R + m_{13}\hat{\mathcal{G}}^A\hat{\sigma}_3\hat{\mathcal{G}}_1^R\right]\\ &= \gamma\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\left[m_{10}\hat{\mathcal{G}}^A\hat{\mathcal{G}}_1^R\hat{\sigma}_0 + m_{11}\hat{\mathcal{G}}^A\hat{\mathcal{G}}_0^R\hat{\sigma}_1 + m_{12}\hat{\mathcal{G}}^A\hat{\mathcal{G}}_3^R\hat{\sigma}_2 + m_{13}\hat{\mathcal{G}}^A\hat{\mathcal{G}}_2^R\hat{\sigma}_3\right]\\ &= m_{10}\hat{m}_1^{(1)}\hat{\sigma}_0 + m_{11}\hat{m}_0^{(1)}\hat{\sigma}_1 + m_{12}\hat{m}_3^{(1)}\hat{\sigma}_2 + m_{13}\hat{m}_2^{(1)}\hat{\sigma}_3.\end{aligned} \tag{A.10}$$

Putting $\hat{m}_\beta^{(1)}$ in (A.10) and collecting the coefficients of each Pauli matrix, we obtain

$$\begin{aligned}\hat{m}_1^{(2)} = {}&\hat{\sigma}_0[(m_{10})^2 + m_{11}m_{01} + m_{12}m_{32} + m_{13}m_{23}] + \hat{\sigma}_1[m_{11}m_{10} + m_{11}m_{00} + \mathrm{i}m_{13}m_{22}]\\ &+ \hat{\sigma}_2[m_{12}m_{10} + m_{12}m_{30} - \mathrm{i}m_{13}m_{21}]\\ &+ \hat{\sigma}_3[m_{13}m_{10} - \mathrm{i}m_{11}m_{02} + \mathrm{i}m_{12}m_{31} + m_{13}m_{20}]\\ &\approx \hat{\sigma}_0[(m_{10})^2] + \hat{\sigma}_1[m_{11}(m_{10}+m_{00})] + \hat{\sigma}_2[m_{12}(m_{10}+m_{30})] + \hat{\sigma}_3[\mathrm{i}m_{12}m_{31}],\end{aligned} \tag{A.11}$$

where the smaller terms are neglected in the last expression. Note that each term in the coefficient of $\hat{\sigma}_3$ is $\mathcal{O}(q^2/k_F^2)$, but the one which is kept has much larger coefficient than the others. Similarly,

$$\begin{aligned}\hat{m}_1^{(3)} \approx {}&\hat{\sigma}_0(m_{10})^3 + \hat{\sigma}_1 m_{11}\left[(m_{10})^2 + (m_{00})^2 + m_{10}m_{00}\right] + \hat{\sigma}_2\left[(m_{10})^2 + (m_{30})^2 + m_{10}m_{30}\right]\\ &+ \hat{\sigma}_3\left[\mathrm{i}m_{12}m_{31}\left(m_{10}+m_{20}+m_{30}\right)\right].\end{aligned} \tag{A.12}$$

One can anticipate the higher orders now. Summing all the orders,

$$\begin{aligned}\sum_{j=0}^{\infty}\hat{m}_1^{(j)} = {}&\hat{\sigma}_0\frac{1}{1-m_{10}} + \hat{\sigma}_1\frac{m_{11}}{(1-m_{10})(1-m_{00})}\\ &+ \hat{\sigma}_2\frac{m_{12}}{(1-m_{10})(1-m_{30})} + \hat{\sigma}_3\frac{\mathrm{i}m_{12}m_{31}}{(1-m_{10})(1-m_{20})(1-m_{30})}.\end{aligned} \tag{A.13}$$

We similarly find

$$\sum_{j=0}^{\infty} \hat{m}_0^{(j)} = \hat{\sigma}_0 \frac{1}{1-m_{00}} + \hat{\sigma}_1 \frac{m_{01}}{(1-m_{00})(1-m_{10})} + \hat{\sigma}_2 \frac{m_{02}}{(1-m_{00})(1-m_{20})} \tag{A.14}$$

$$\sum_{j=0}^{\infty} \hat{m}_3^{(j)} = \hat{\sigma}_0 \frac{1}{1-m_{30}} + \hat{\sigma}_1 \frac{m_{31}}{(1-m_{20})(1-m_{30})} + \hat{\sigma}_2 \frac{m_{32}}{(1-m_{10})(1-m_{30})} \tag{A.15}$$

$$\begin{aligned}\sum_{j=0}^{\infty} \hat{m}_2^{(j)} = {}& \hat{\sigma}_0 \frac{1}{1-m_{20}} + \hat{\sigma}_1 \frac{m_{21}}{(1-m_{20})(1-m_{30})} \\ & + \hat{\sigma}_2 \frac{m_{22}}{(1-m_{20})(1-m_{00})} + \hat{\sigma}_3 \frac{-\mathrm{i}m_{21}m_{32}}{(1-m_{10})(1-m_{20})(1-m_{30})}.\end{aligned} \tag{A.16}$$

It is now straightforward to calculate $\chi_{\alpha\beta}$ using equations (A.8), and (A.13)–(A.16):

$$\hat{\chi}(\mathbf{q},\omega) = 2\nu\mathrm{i}\omega\tau \begin{bmatrix} \frac{m_{00}}{1-m_{00}} & \frac{m_{11}}{(1-m_{00})(1-m_{10})} & \frac{m_{22}}{(1-m_{00})(1-m_{20})} & 0 \\ \frac{m_{01}}{(1-m_{00})(1-m_{10})} & \frac{m_{10}}{1-m_{10}} & -\mathrm{i}\frac{m_{21}m_{32}}{(1-m_{10})(1-m_{20})(1-m_{30})} & \mathrm{i}\frac{m_{32}}{(1-m_{10})(1-m_{30})} \\ \frac{m_{02}}{(1-m_{00})(1-m_{20})} & \mathrm{i}\frac{m_{12}m_{31}}{(1-m_{10})(1-m_{20})(1-m_{30})} & \frac{m_{20}}{1-m_{20}} & -\mathrm{i}\frac{m_{31}}{(1-m_{20})(1-m_{30})} \\ 0 & -\mathrm{i}\frac{m_{12}}{(1-m_{10})(1-m_{30})} & \mathrm{i}\frac{m_{21}}{(1-m_{20})(1-m_{30})} & \frac{m_{30}}{1-m_{30}} \end{bmatrix} \tag{A.17}$$

The diffusive propagator $\mathcal{D}_{\alpha\beta}$ is connected with the charge–spin density correlation function by the relation

$$\chi_{\alpha\beta} = 2\nu\mathrm{i}\omega\tau(D_{\alpha\beta} - \delta_{\alpha\beta}) \tag{A.18}$$

We thus find the inverse diffusive propagator to be

$$\mathcal{D}^{-1} = \begin{bmatrix} 1-m_{00} & -m_{11} & -m_{22} & 0 \\ -m_{01} & 1-m_{10} & \frac{m_{21}m_{32}}{1-m_{30}} & -\mathrm{i}m_{32} \\ -m_{02} & \frac{m_{12}m_{31}}{1-m_{30}} & 1-m_{20} & \mathrm{i}m_{31} \\ 0 & \mathrm{i}m_{12} & -\mathrm{i}m_{21} & 1-m_{30} \end{bmatrix}. \tag{A.19}$$

Instead of calculating $\hat{\chi}(\mathbf{q},\omega)$ explicitly as we have done above (A.17), if one defines

$$\chi_{\alpha\beta}(\mathbf{q},\omega) = \mathrm{i}\nu\omega\tau\mathcal{I}_{\alpha\delta}\overline{\mathcal{D}}_{\delta\beta} \tag{A.20}$$

with

$$\mathcal{I}_{\alpha\beta} = \frac{\gamma}{2}\,\mathrm{Tr}\left[\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\hat{\sigma}_\alpha \hat{\mathcal{G}}^A(\mathbf{k}',0)\hat{\sigma}_\beta\hat{\mathcal{G}}^R(\mathbf{k}'+\mathbf{q},\omega)\right] \tag{A.21}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\sigma_\alpha \hat{m}_\beta^{(1)}\sigma_\beta\right] \tag{A.22}$$

and

$$\overline{\mathcal{D}}_{\alpha\beta}^{-1} = \delta_{\alpha\beta} - \mathcal{I}_{\alpha\beta} \tag{A.23}$$

as in [24], one finds the inverse diffusive propagator to be

$$\overline{\mathcal{D}}^{-1} = \begin{bmatrix} 1-m_{00} & -m_{11} & -m_{22} & 0 \\ -m_{01} & 1-m_{10} & \mathrm{i}m_{23} & -\mathrm{i}m_{32} \\ -m_{02} & -\mathrm{i}m_{13} & 1-m_{20} & \mathrm{i}m_{31} \\ 0 & \mathrm{i}m_{12} & -\mathrm{i}m_{21} & 1-m_{30} \end{bmatrix}. \tag{A.24}$$

Clearly, $\mathcal{D}_{12}^{-1} \neq \overline{\mathcal{D}}_{12}^{-1}$ and $\mathcal{D}_{21}^{-1} \neq \overline{\mathcal{D}}_{21}^{-1}$ but the other components of $\mathcal{D}^{-1}$ and $\overline{\mathcal{D}}^{-1}$ are the same. While $\overline{\mathcal{D}}_{12}^{-1}$ and $\overline{\mathcal{D}}_{21}^{-1}$ are very small since m_{13} and m_{23} are small due to their quadratic dependence on the small parameter δ, $\mathcal{D}_{12}^{-1}$ and $\mathcal{D}_{21}^{-1}$ are not small because these are independent of δ in the limit of small δ. These are indeed the important terms for transverse spin diffusion and hence spin–spin Hall effect. The way of obtaining inverse diffusive propagator using combined expressions (A.20)–(A.23) is incorrect because the non-commutative nature of Pauli matrices is not correctly taken care of, as we have seen from our explicit calculation of $\hat{\chi}$ in equation (A.17).

Appendix B. Role of Dresselhaus spin–orbit coupling

Due to the bulk inversion asymmetry, the 2DES will also have Dresselhaus spin–orbit coupling. The strength of this coupling is however much smaller than the Rashba coupling. This appendix contains the derivation of the dc spin–spin Hall conductivities in the presence of both Rashba and Dresselhaus spin–orbit couplings. The Hamiltonian in this case is

$$H = H_0 + V(\mathbf{r}); \qquad H_0 = \frac{\mathbf{p}^2}{2m} + \lambda\left(\hat{\sigma}_1 p_y - \hat{\sigma}_2 p_x\right) + \beta\left(\hat{\sigma}_1 p_x - \hat{\sigma}_2 p_y\right) \tag{B.1}$$

with the eigenvalues

$$\epsilon_k^{\mathrm{s}} = \frac{\mathbf{k}^2}{2m} + s\lambda_{\mathrm{RD}} k; \qquad \lambda_{\mathrm{RD}}(\theta) = \sqrt{\lambda^2 + \beta^2 + 4\lambda\beta\sin\theta\cos\theta} \tag{B.2}$$

where β is the Dresselhaus SO coupling strength. The energy spectrum now depends on modified SO coupling strength $\lambda_{\mathrm{RD}}(\theta)$. The Fermi surfaces of both the branches are anisotropic here and the corresponding Fermi momenta are $k_{\mathrm{F}}^{\mathrm{s}} = -sm\lambda_{\mathrm{RD}} + \sqrt{k_{\mathrm{F}}^2 + m^2\lambda_{\mathrm{RD}}^2}$.

The retarded (advanced) Green's functions for the electrons in this case can be written as

$$\hat{\mathcal{G}}^{R,A}(\mathbf{k},\epsilon) = \frac{1}{2}\sum_{s=\pm}\frac{\hat{\sigma}_0 + s\left(\cos\hat{\chi}_{\mathbf{k}}\hat{\sigma}_1 - \sin\hat{\chi}_{\mathbf{k}}\hat{\sigma}_2\right)}{\epsilon - \xi_{\mathbf{k}}^{\mathrm{s}} \pm \mathrm{i}/2\tau} \tag{B.3}$$

where $\cos\hat{\chi}_{\mathbf{k}} = \frac{\lambda\sin\theta+\beta\cos\theta}{\lambda_{\mathrm{RD}}}$ and $\sin\hat{\chi}_{\mathbf{k}} = \frac{\lambda\cos\theta+\beta\sin\theta}{\lambda_{\mathrm{RD}}}$.

We proceed to calculate spin–spin Hall conductivities in the same way as in section 3, but the calculation is much more cumbersome due to the angle dependent spin–orbit coupling strength $\lambda_{\mathrm{RD}}(\theta)$ in the dispersion. We find however that one can avoid a lot of complication in self-consistent evaluation of the vertex corrected currents, by concentrating on the largest order in Δ. As long as the conductivities are non-zero in this order, the lower orders may be ignored. The vertex-corrected in-plane spin current components in the dc limit are thus found to be

$$\hat{J}_1^x = \hat{J}_2^y \approx -\frac{2\Delta}{B}\frac{\lambda B + \beta\left(\sqrt{A^2-B^2} - A\right)}{\sqrt{A^2-B^2} - 1}\hat{\sigma}_3 \tag{B.4}$$

$$\hat{J}_2^x = \hat{J}_1^y \approx -\frac{2\Delta}{B}\frac{\beta B + \lambda\left(\sqrt{A^2-B^2} - A\right)}{\sqrt{A^2-B^2} - 1}\hat{\sigma}_3 \tag{B.5}$$

with $A = 1 + 4\delta^2 + 4\delta_D^2$, $B = 8\delta\delta_D$, and $\delta_D = \beta k_{\mathrm{F}}\tau$.

The dc spin–spin Hall conductivities, defined in equations (5) and (6), can be obtained now via retarded current–current correlation function (7). We find

$$\sigma_{xy}^{\perp} \approx -\frac{\Delta}{8\pi}\left(\frac{A^2 - B^2 - A}{B^2}\right)\frac{\sqrt{A^2-B^2} - A}{\sqrt{A^2-B^2} - 1} \tag{B.6}$$

$$\sigma_{xy}^{\parallel} \approx -\frac{\Delta}{8\pi}\left(\frac{1}{B}\right)\frac{\sqrt{A^2-B^2} - A}{\sqrt{A^2-B^2} - 1}. \tag{B.7}$$

The fact that $\sigma_{xy}^{\parallel}$ is non-zero suggests that the spin polarization of the induced spin current in the transverse direction is not fully perpendicular to the applied spin current. It is, in fact, fully perpendicular when one of the couplings is absent. Therefore, in realistic situations, where $\beta \ll \lambda$, the spin polarizations of the applied and induced spin currents are almost perpendicular to each other as $\sigma_{xy}^{\parallel}/\sigma_{xy}^{\perp} \sim \delta_D/[\delta(1+4\delta^2)] \ll 1$ in this limit.

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